B2-Topology-Midsem Test-2025

Instructions: Total time 3 Hours. **All questions are compulsory**. All questions carry 6 marks. You may use results proved in the class without proof. Use concepts, notations, terminology, results, as covered in the course. If you wish to use a problem from a homework/assignment, supply its solution too.

1. Prove that a connected metric space with at least two points is uncountable.
2. Let $n \geq 1$ and $S^n \subset \mathbb{R}^{n+1}$ denote the n-sphere given by $S^n = \{(x_0, \cdots, x_n) \in \mathbb{R}^{n+1} | x_0^2 + \cdots + x_n^2 = 1\}$. Let $D \subset \mathbb{R}^{n+1}$ be a countably infinite subset with the property that $d := \inf\{\|x - y\|,\ x \neq y,\ x, y \in D\} > 0$. Prove that $D \cap S^n$ must be finite.
3. Let X, Y be metric spaces and $f : X \to Y$ be a continuous map. Let $D \subset X$ be dense. Prove that f is uniquely determined by its values on D.
4. Recall that a point p in a metric space X is called an isolated point, if for some $\epsilon > 0$, $B(p, \epsilon) \cap X = \{p\}$, i.e. $\{p\}$ is an open subset of X. Let Y be a countable complete metric space. Prove that Y has an isolated point.
5. Prove that $\mathrm{GL}(n, \mathbb{C})$ is path-connected, where we treat $\mathrm{GL}(n, \mathbb{C})$ as a metric subspace of $\mathbb{C}^{n^2}$.
6. Let X be a compact metric space, $x \in X$ and $C \subset X$ be a closed set not containing x. Prove that there are disjoint open subsets U, V of X, with $x \in U$, $C \subset V$.